Topic: Sequence and Series

Introduction:

Important Concepts and Formulas - Sequence and Series

Arithmetic Progression (AP) :

Arithmetic progression (AP) or arithmetic sequence is a sequence of numbers in which each term after the first is obtained by adding a constant, d to the preceding term. The constant d is called common difference.

An arithmetic progression is given by a, (a + d), (a + 2d), (a + 3d), ..

where a = the first term , d = the common difference

Examples :

1, 3, 5, 7, ... is an arithmetic progression (AP) with a = 1 and d = 2

7, 13, 19, 25, ... is an arithmetic progression (AP) with a = 7 and d= 6

If a, b, c are in AP, 2b = a + c

n^{th} term of an arithmetic progression

$t_n = a + (n - 1)d$

where t_n = n^{th} term, a= the first term , d= common difference

Example 1

Find 10^{th} term in the series 1, 3, 5, 7, ...

a = 1

d = 3 – 1 = 2

10^{th} term, t10 = a + (n-1)d = 1 + (10 – 1)2 = 1 + 18 = 19

Example 2

Find 16^{th} term in the series 7, 13, 19, 25, ...

a = 7

d = 13 – 7 = 6

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16th term, t_{16} = a + (n-1)d = 7 + (16 – 1)6 = 7 + 90 = 97

Number of terms of an arithmetic progression

$$n=\frac{(l-a)}{d}+1=\frac{(72-8)}{4}+1$$

= 64/4 +1 = 16 +1 =17

Sum of first n terms in an arithmetic progression

Sn=n/2[2a+(n−1)d] =n/2(a+l)

where a = the first term,

d= common difference,

l = t_n = n^{th} term = a + (n-1) d

Example

Find 4 + 7 + 10 + 13 + 16 + . . . up to 20 terms

a = 4

d = 7 – 4 = 3

Sum of first 20 terms, S_{20}

= n/2[2a+(n−1)d]

= 20/2[(2 × 4) + (20-1)3]

=10[8+(19×3)]

=10(8+57)

=650

Arithmetic Mean

If a, b, c are in AP, b is the Arithmetic Mean (AM) between a and c. In this case, b=1/2(a+c)
The Arithmetic Mean (AM) between two numbers a and b =1/2(a+c)

If a, a_1, a_2 ... a_n, b are in AP we can say that a_1, a_2 ... a_n are the n Arithmetic Means

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between a and b.

Additional Notes on AP

To solve most of the problems related to AP, the terms can be conveniently taken as

3 terms: (a – d), a, (a +d)

4 terms: (a – 3d), (a – d), (a + d), (a +3d)

5 terms: (a – 2d), (a – d), a, (a + d), (a +2d)

$T_n = S_n - S_{n-1}$

If each term of an AP is increased, decreased , multiplied or divided by the same non-zero constant, the resulting sequence also will be in AP.

In an AP, sum of terms equidistant from beginning and end will be constant.

Harmonic Progression (HP)

Non-zero numbers a1, a2, a3, ⋯ ana1, a2, a3, ⋯ an are in Harmonic Progression(HP) if $\frac{1}{a1}, \frac{1}{a2}, \frac{1}{a3}, \ldots\frac{1}{an}$ are in AP. Harmonic Progression is also known as harmonic sequence.

Examples

$\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \cdots$ is a harmonic progression (HP)

Three non-zero numbers a, b, c will be in HP, if $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in AP.

If a, (a+d), (a+2d), . . . are in AP, n^{th} term of the AP = a + (n - 1)d

Hence, if $\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d},\cdots$ are in HP, n^{th} term of the HP = $\frac{1}{a+(n-1)d}$

If a, b, c are in HP, b is the Harmonic Mean(HM) between a and c

In this case, b= $\frac{2ac}{a+c}$

The Harmonic Mean(HM) between two numbers a and b = $\frac{2ab}{a+b}$

If a, a_1, a_2 ... a_n, b are in HP we can say that a_1, a_2 ... a_n are the n Harmonic Means between a and b.

If a, b, c are in HP, $\frac{2}{b} = \frac{1}{a},\frac{1}{c}$

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Geometric progression(GP)

Geometric Progression(GP) or Geometric Sequence is sequence of non-zero numbers in which the ratio of any term and its preceding term is always constant.

A geometric progression(GP) is given by a, ar, ar^2, ar^3, ...

where a = the first term , r = the common ratio

Examples

1, 3, 9, 27, ... is a geometric progression(GP) with a = 1 and r = 3

2, 4, 8, 16, ... is a geometric progression(GP) with a = 2 and r = 2

If a, b, c are in GP, $b^2 = ac$

n^{th} term of a geometric progression(GP)

$t_n=ar^{n-1}$

where t_n = n^{th} term, a= the first term , r = common ratio, n = number of terms

Example 1
Find the 10^{th} term in the series 2, 4, 8, 16, ...
a = 2, r =4/2= 2, n = 10

10th term, t_{10}

$=ar^{n-1}=2\times2^{10-1}=2\times2^9=2\times512=1024$

Sum of first n terms in a geometric progression(GP)

$$S_n = \begin{cases} \dfrac{a(r^n-1)}{r-1} & (\text{if } r>1) \\ \dfrac{a(1-r^n)}{1-r} & (\text{if } r<1) \end{cases}$$

where a= the first term,
r = common ratio,

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n = number of terms

Example 1

Find 4 + 12 + 36 + ... up to 6 terms

$$a = 4, \quad r = \frac{12}{4} = 3, \quad n = 6$$

Here r > 1. Hence,

$$S_6 = \frac{a(r^n - 1)}{r - 1} = \frac{4(3^6 - 1)}{3 - 1}$$

$$= \frac{4(729 - 1)}{2} = \frac{4 \times 728}{2}$$

$$= 2 \times 728 = 1456$$

Sum of an infinite geometric progression(GP)

$S_\infty = \frac{a}{1 - r}$ (if $-1 < r < 1$)

where a= the first term , r = common ratio

Example

$$\text{Find } 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \infty$$

$$a = 1, \quad r = \frac{\left(\frac{1}{2}\right)}{1} = \frac{1}{2}$$

Here -1 < r < 1. Hence,

$$S_\infty = \frac{a}{1 - r} = \frac{1}{\left(1 - \frac{1}{2}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

Geometric Mean

If three non-zero numbers a, b, c are in GP, b is the Geometric Mean(GM) between a and c. In this case, $b = \sqrt{ac}$

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The Geometric Mean(GM) between two numbers a and b = √ab

(Note that if a and b are of opposite sign, their GM is not defined.)

Additional Notes on GP

To solve most of the problems related to GP, the terms of the GP can be conveniently taken as

3 terms : $\frac{a}{r}$, a, ar

5 terms: a/r^2, $\frac{a}{r}$, a, ar, ar^2

If a, b, c are in GP = $\frac{a-b}{b-c} = \frac{a}{b}$

In a GP, product of terms equidistant from beginning and end will be constant.

Relationship Between Arithmetic Mean, Harmonic Mean, and Geometric Mean of Two Numbers

If GM, AM and HM are the Geometric Mean, Arithmetic Mean and Harmonic Mean of two positive numbers respectively, then

$GM^2 = AM \times HM$

Some Interesting Properties to Note

Three numbers a, b and c are in AP if b = $\frac{a+c}{2}$

Three non-zero numbers a, b and c are in HP if b = $\frac{2ac}{a+c}$

Three non-zero numbers a, b and c are in HP if $\frac{a-b}{b-c} = \frac{a}{c}$

Let A, G and H be the AM, GM and HM between two distinct positive numbers. Then

(1) A > G > H

(2) A, G and H are in GP

If a series is both an AP and GP, all terms of the series will be equal. In other words, it will be a constant sequence. Let A, G and H be the AM, GM and HM between two distinct positive numbers. Then

(1) A > G > H

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(2) A, G and H are in GP
If a series is both an AP and GP, all terms of the series will be equal. In other words, it will be a constant sequence.

Power Series : Important formulas

$1+1+1+\cdots$ n terms $= \sum 1 = n$

$1+2+3+\cdots+n = \sum n = \frac{n(n+1)}{2}$

$1^2+2^2+3^2+\cdots+n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$

$1^3+2^3+3^3+\cdots+n^3 = \sum n^3 =$

$$\frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2$$