Topic: Quadratic Equations

Introduction:

In algebra, a **quadratic equation** (from the Latin *quadratus* for "square") is any equation having the form where *x* represents an unknown, and *a*, *b*, and *c* represent known numbers such that *a* is not equal to 0. If *a* = 0, then the equation is linear, not quadratic. The numbers *a*, *b*, and *c* are the *coefficients* of the equation, and may be distinguished by calling them, respectively, the *quadratic coefficient*, the *linear coefficient* and the *constant* or *free term*. Because the quadratic equation involves only one unknown, it is called "univariate". The quadratic equation only contains powers of *x* that are non-negative integers, and therefore it is a polynomial equation, and in particular it is a second degree polynomial equation since the greatest power is two.

Relation Between Roots & Co-efficients:

(i) The solutions of quadratic equation, $ax^2 + bx + c = 0$, $(a \neq 0)$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(ii) The expression, $b2 - 4ac \equiv D$ is called discriminant of quadratic equation. If α, β are the roots of quadratic equation, $a x 2 + b x + c = 0$, $a \neq 0$. Then:

(a) $\alpha + \beta = -b/a$ (b) $\alpha\beta = c/a$ (c) $|\alpha - \beta| = \frac{\sqrt{D}}{|a|}$

A quadratic equation whose roots are α & β, is $(x - \alpha)(x - \beta) = 0$ i.e.

x^2 – (sum of roots) x + (product of roots) = 0

Example 2: If α and β are the roots of $ax^2 + bx + c = 0$, find the equation whose roots are $\alpha+2$ and $\beta+2$. Solution. Replacing x by x – 2 in the given equation, the required equation is $a(x - 2)2 + b(x - 2) + c = 0$ i.e., $ax^2 - (4a - b)x + (4a - 2b + c) = 0$.

Example 3: The coefficient of x in the quadratic equation $x^2 + px + q = 0$ was taken as 17 in place of 13, its roots were found to be – 2 and – 15. Find the roots of the original equation.

Solution. Here $q = (-2) \times (-15) = 30$, correct value of p = 13. Hence original equation is $x^2 + 13x + 30 = 0$ as $(x + 10)(x + 3) = 0$ $\therefore$ roots are – 10, – 3

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Equation v/s Identity:

A quadratic equation is satisfied by exactly two values of ' x ' which may be real or imaginary. The equation, a x 2 + b x + c = 0 is:

↔ a quadratic equation if a ≠ 0 Two Roots

↔ a linear equation if a = 0, b ≠ 0 One Root

↔ a contradiction if a = b = 0, c ≠ 0 No Root

↔ an identity if a = b = c = 0 Infinite Roots If a quadratic equation is satisfied by three distinct values of ' x ', then it is an identity.

Example 1:

(i) 3x2 + 2x – 1 = 0 is a quadratic equation here a = 3.

(ii) (x + 1)2 = x2 + 2x + 1 is an identity in x.

Solution:

Here highest power of x in the given relation is 2 and this relation is satisfied by three different values x= 0, x = 1 and x = – 1 and hence it is an identity because a polynomial equation of nth degree cannot have more than n distinct roots.

Before proceeding with this section we should note that the topic of solving quadratic equations will be covered in two sections. This is done for the benefit of those viewing the material on the web. This is a long topic and to keep page load times down to a minimum the material was split into two sections.

So, we are now going to solve quadratic equations. First, the **standard form** of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

$ax^2 + bx + c = 0 \qquad a \neq 0$

The only requirement here is that we have an x^2 x^2 in the equation. We guarantee that this term will be present in the equation by requiring $a \neq 0$ $a \neq 0$. Note however, that it is okay if b and/or c are zero.

There are many ways to solve quadratic equations. We will look at four of them over the course of the next two sections. The first two methods won't always work, yet are probably a little simpler to use when they work. This section will cover these two

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methods. The last two methods will always work, but often require a little more work or attention to get correct. We will cover these methods in the next section.

So, let’s get started.

Solving by Factoring:
As the heading suggests we will be solving quadratic equations here by factoring them. To do this we will need the following fact.

$$\text{If } ab = 0 \quad \text{then either} \quad a =$$

$$\text{If } ab = 0 \quad \text{then either } a = 0 \text{ and/or } b$$

This fact is called the **zero factor property** or **zero factor principle**. All the fact says is that if a product of two terms is zero then at least one of the terms had to be zero to start off with.

Notice that this fact will ONLY work if the product is equal to zero. Consider the following product.

$$ab = 6 \qquad ab = 6$$

In this case there is no reason to believe that either a or b will be 6. We could have $a = 2$ $a = 2$ and $b = 3$ $b = 3$ for instance. So, do not misuse this fact!

To solve a quadratic equation by factoring we first must move all the terms over to one side of the equation. Doing this serves two purposes. First, it puts the quadratics into a form that can be factored. Secondly, and probably more importantly, in order to use the zero factor property we MUST have a zero on one side of the equation. If we don’t have a zero on one side of the equation we won’t be able to use the zero factor property.

Read more:

(i) If α is a root of the equation f(x) = 0, then the polynomial f(x) is exactly divisible by (x – α) or (x – α) is a factor of f(x) and conversely.

(ii) Every equation of nth degree ($n \geq 1$) has exactly n roots & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation f(x) = 0 are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. imaginary roots occur in conjugate pairs.

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(iv) An equation of odd degree will have odd number of real roots and an equation of even degree will have even numbers of real roots.

(v) If the coefficients in the equation are all rational & $\alpha + \beta$ is one of its roots, then $\alpha - \beta$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(vi) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x) = 0 must have odd number of real roots (also atleast one real root) between ' a ' and ' b '.

(vii) Every equation f(x) = 0 of degree odd has atleast one real root of a sign opposite to that of its last term. (If coefficient of highest degree term is positive).

NATURE OF ROOTS:

(A) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then ;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal).

(iii) $D < 0 \Leftrightarrow$ roots are imaginary .

(iv) If $p + i q$ is one root of a quadratic equation, then the other must be the conjugate $p - i q$ & vice versa. ($p , q \in R$ & $i = \sqrt{-1}$).

(B) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then;

(i) If D > 0 & is a perfect square , then roots are rational & unequal.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then the other root must be the conjugate of it i.e. $\beta = p - \sqrt{q}$ & vice versa.

Note :

(i) If α is a root of the equation f(x) = 0, then the polynomial f(x) is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of f(x) and conversely .

(ii) Every equation of nth degree ($n \geq 1$) has exactly n roots & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation f(x) = 0 are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$

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is also a root. i.e. imaginary roots occur in conjugate pairs.

(iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x) = 0 must have atleast one real root between 'a' and 'b' .

(vi) Every equation f(x) = 0 of degree odd has atleast one real root of a sign opposite to that of its last term.

LOCATION OF ROOTS :

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

(i) Conditions for both the roots of f (x) = 0 to be greater than a specified number 'd' are b 2 – 4ac ≥ 0; f(d) > 0 & (–b/2a) > d.

(ii) Conditions for both roots of f(x) = 0 to lie on either side of the number 'd' (in other words the number 'd' lies between the roots of f(x) = 0) is f(d) < 0.

(iii) Conditions for exactly one root of f(x) = 0 to lie in the interval (d, e) i.e. d < x < e are $b^2 - 4ac > 0$ & f(d) .f(e) < 0.

(iv) Conditions that both roots of f (x) = 0 to be confined between the numbers p & q are (p < q). $b^2 - 4ac \geq 0$; f(p) > 0; f(q) > 0 & p < (– b/2a) < q.

LOGARITHMIC INEQUALITIES:

(i) For a > 1 the inequality 0 < x < y & $\log_a x < \log_a y$ are equivalent.

(ii) For 0 < a < 1 the inequality 0 < x < y & $\log_a x > \log_a y$ are equivalent.

(iii) If a > 1 then $\log_a x < p \Rightarrow 0 < x < a^p$

(iv) If a > 1 then $\log_a x > p \Rightarrow x > a^p$

(v) If 0 < a < 1 then $\log_a x < p \Rightarrow x > a^p$

(vi) If 0 < a < 1 then $\log_a x > p \Rightarrow 0 < x < a^p$