Topic: Functions and Inequalities

Introduction:

Functions :

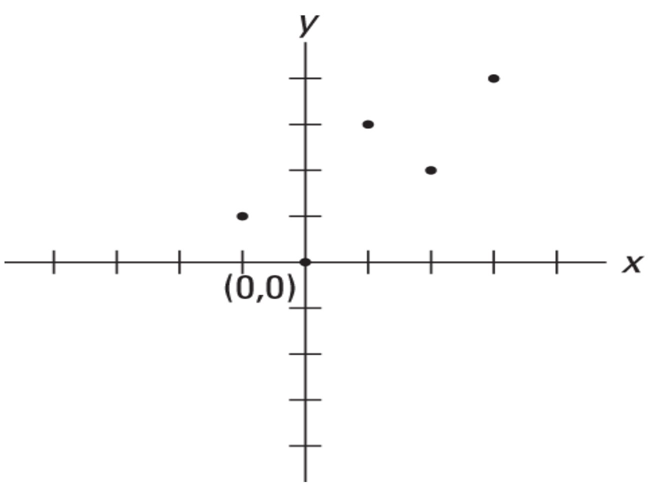
Functions are very specific types of relations. Before defining a function, it is important to define a relation.

Relations:

Any set of ordered pairs is called a **relation.** Figure 1 shows a set of ordered pairs.

$A = \{(-1, 1), (1, 3), (2, 2), (3, 4)\}$

Figure 1. A graph of the set of ordered pairs (–1, 1), (1, 3), (2, 2), (3, 4).



Domain and range:

The set of all *x*'s is called the domain of the relation. The set of all *y*'s is called the range of the relation. The domain of set *A* in Figure 1 is {–1, 1, 2, 3}, while the range of set *A* is {1, 2, 3, 4}.

Example 1: Find the domain and range of the set of graphed points in Figure 2.

The domain is the set {–2, –1, 1, 3}. The range is the set {–1, 2, 3}.

Figure 2. Plotted points.

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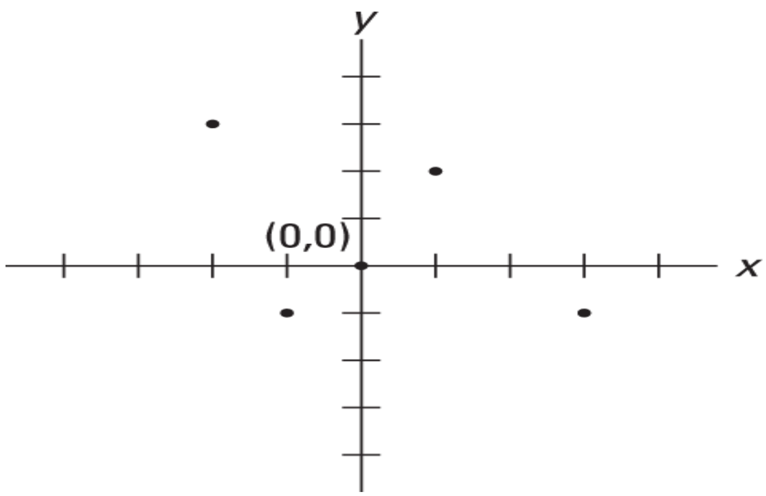

Defining a function:

The relation in Example has pairs of coordinates with unique first terms. When the *x* value of each pair of coordinates is different, the relation is called a *function.* A function is a relation in which each member of the domain is paired with exactly one element of the range. *All functions are relations, but not all relations are functions.* A good example of a functional relation can be seen in the linear equation $y = x + 1$, graphed in Figure 3. The domain and range of this function are both the set of real numbers, and the relation is a function because for any value of *x* there is a unique value of *y.*

Figure 3. A graph of the linear equation $y = x + 1$.

Graphs of functions:

In each case in Figure 4 (a), (b), and (c), for any value of *x*, there is only one value for *y.* Contrast this with the graphs in Figure 5.

Figure 4. Graphs of functions.

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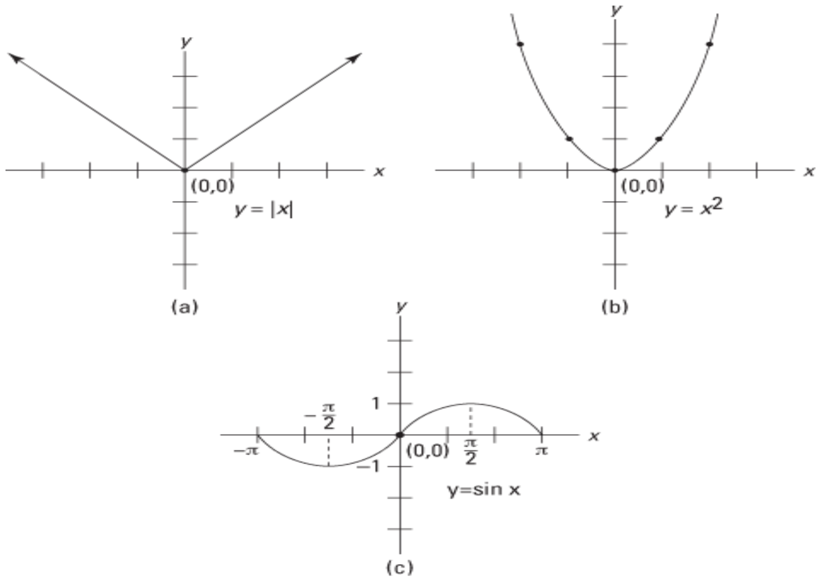
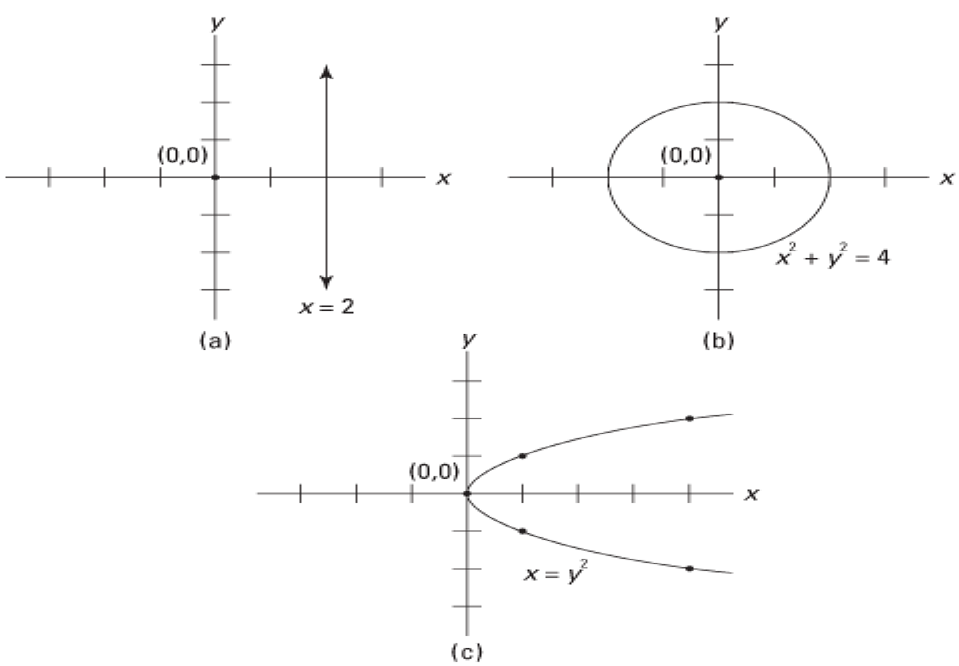


Graphs of relationships that are not functions:

In each of the relations in Figure 5 (a), (b), and (c), a single value of *x* is associated with two or more values of *y*. These relations are not functions.

Figure 5. Graphs of relations that are not functions.



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Determining domain, range, and if the relation is a function
Example 2

1.

$B = \{(-2, 3)(-1, 4)(0, 5)(1, -3)\}$ domain: $\{-2, -1, 0, 1\}$

range: $\{-3, 3, 4, 5\}$

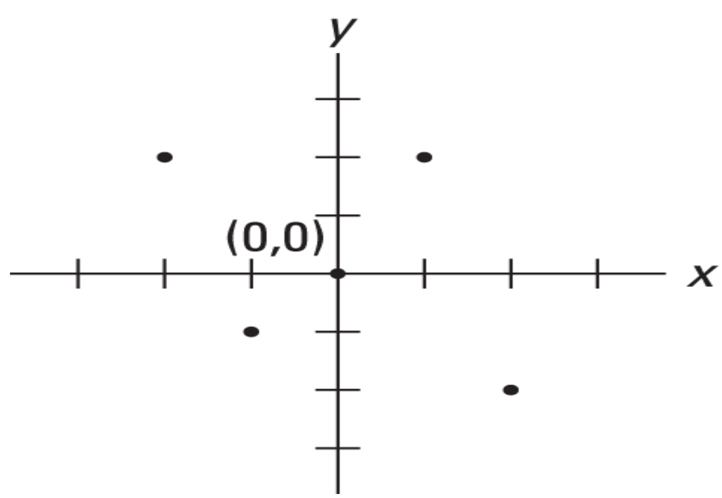
function: yes

2.

domain: $\{-2, -1, 1, 2\}$

range: $\{-2, -1, 2\}$

function: yes



3.

domain: $\{x: x \leq 1\}$

range: $\{y: y \geq -3\}$

function: yes

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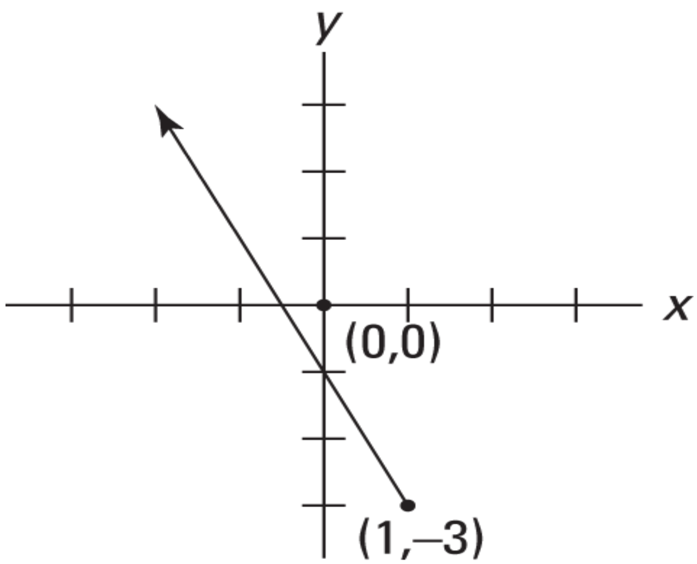
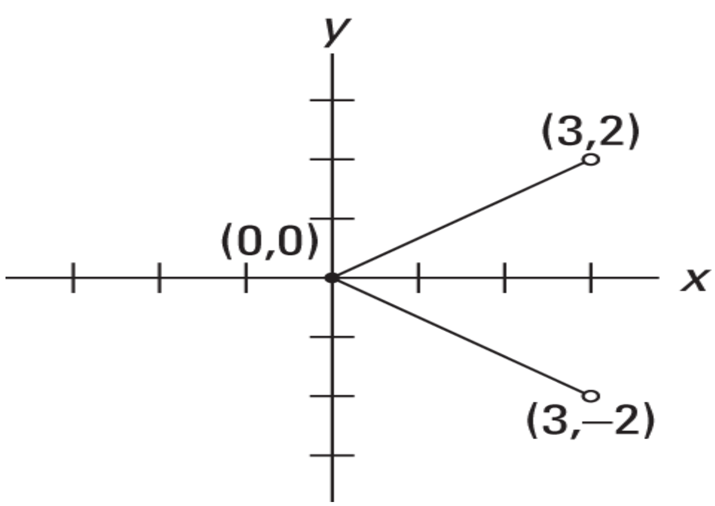


4.

domain: $\{x:\ 0 \le x < 3\}$

range: $\{y:\ -2 < y < 2\}$

function: no



5.

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$$y = x^2 \quad \text{domain:}\{\text{all real numbers}\}$$
$$\text{range:}\{y: y \geq 0\}$$
$$\text{function: yes}$$

6.

$$x = y^2 \quad \text{domain:}\{x: x \geq 0\}$$
$$\text{range:}\{\text{all real numbers}\}$$
$$\text{function: no}$$

Note that Examples 2(e) and (f) are illustrations of *inverse relations*: relations in which the domain and the range have been interchanged. Notice that while the relation in (e) is a function, the inverse relation in (f) is not.

The graph of a function f is the set of all points in the plane of the form (x,f(x)). We could also define the graph off to be the graph of the equation y=f(x). So the graph of a function is a special case of the graph of an equation.
Example:

Let $f(x)=x^2-3$

Recall that when we introduced graphs of equations we noted that if we can solve the equation for y, then it is easy to find points that are on the graph. We simply choose a number for x, then compute the corresponding value of y. Graphs of functions are graphs of equations that have been solved for y!
The graph of f(x) in this example is the graph of $y=x^2-3$. It is easy to generate points on the graph. Choose a value for the first coordinate, then evaluate f at that number to find the second coordinate. The following table shows several values for xx and the function f evaluated at those numbers.

x	-2	-1	0	1	2
$f(x)$	1	-2	-3	-2	1

Each column of numbers in the table holds the coordinates of a point on the graph.

Example:

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Let ff be the piecewise-defined function

$$f(x) = \begin{cases} 5 - x^2 & x \leq 2 \\ x - 1 & x > 2 \end{cases}$$

Some of the Important Functions :

Function	Definition	Example
Linear	The highest power over the x variable is 1	$y = 2x + 1$
Quadratic	The highest power over the x variable(s) is 2	$y = -2x^2 - 7x + 2$
Cubic	The highest power over the x variable(s) is 3	$y = x^3 + 2x^2 + x$
Square Root	The x variable is square rooted	$y = -\sqrt{x - 1}$
Absolute Value	The x variable is within absolute value signs	$y = -\lvert x + 1 \rvert$
Rational	The x variable is in the denominator $(x \neq 0)$	$y = \frac{3}{x}$
Logarithmic	The log of the x variable is taken	$\ln(x)$

Finding the values of functions:

The *value of a function* is really the *value of the range* of the relation. Given the function

f = {(1, −3), (2, 4), (−1, 5), (3, −2)}

the value of the function at 1 is −3, at 2 is 4, and so forth. This is written *f*(1) = −3 and *f*(2) = 4 and is usually read, " *f* of 1 = −3 and *f* of 2 = 4." The lowercase letter *f* has been used here to indicate the concept of function, but any lowercase letter might have been used.

Even and Odd Functions

A function f is even if its graph is symmetric with respect to the y-axis. This criterion can be stated algebraically as follows: f is even if f(−x)=f(x) for all xx in the domain of f.

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Example:

If you evaluate f at 3 and at -3, then you will get the same value if f is even.

A function ff is odd if its graph is symmetric with respect to the origin. This criterion can be stated algebraically as follows: ff is odd if $f(-x)=-f(x)$ for all x in the domain of f.

Example:

If you evaluate f at 3, you get the negative of $f(-3)$ when f is odd.

Inequalities:

An inequality is a relation that holds between two values when they are different (see also: equality). The notation $a \neq b$ means that a is not equal to b. It does not say that one is greater than the other, or even that they can be compared in size.

If the values in question are elements of an ordered set, such as the integers or the real numbers, they can be compared in size. The notation $a < b$ means that a is less than b.The notation $a > b$ means that a is greater than b.

In either case, a is not equal to b. These relations are known as strict inequalities. The notation $a < b$ may also be read as "a is strictly less than b".

In contrast to strict inequalities, there are two types of inequality relations that are not strict:The notation $a \leq b$ means that a is less than or equal to b (or, equivalently, not greater than b, or at most b); "not greater than" can also be represented by the symbol for "greater than" bisected by a vertical line, "not."

The notation $a \geq b$ means that a is greater than or equal to b (or, equivalently, not less than b, or at least b),; "not less than" can also be represented by the symbol for "less than" bisected by a vertical line, "not."

In engineering sciences, a less formal use of the notation is to state that one quantity is "much greater" than another, normally by several orders of magnitude.

The notation $a \ll b$ means that a is much less than b. (In measure theory, however, this notation is used for absolute continuity, an unrelated concept.)

The notation $a \gg b$ means that a is much greater than b. An inequality is the result of replacing the "==" sign in an equation with ,≤, or ≥,≤, or ≥.

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For example, 3x−20, are referred to as polynomial inequalities, or quadratic inequalities if the degree is exactly 2.

Inequalities involving rational expressions are called rational inequalities. Some often used inequalities also involve absolute value expressions.

Solving Inequalities

In a nutshell, solving inequalities is about one thing: sign changes. Find all the points at which there are sign changes - we call these points' critical values.

Then determine which, if any, of the intervals bounded by these critical values result in a solution.The solution to the inequality will consist of the set of all points contained by the solution intervals.

Method to Solve Linear, Polynomial or Absolute Value Inequalities:

1. Move all terms to one side of the inequality sign by applying the Addition, Subtraction, Multiplication, and Division Properties of Inequalities. You should have only zero on one side of the inequality sign.

2. Solve the associated equation using an appropriate method. This solution or solutions will make up the set of critical values. At these values, sign changes occur in the inequality.

3. Plot the critical values on a number line. Use closed circles ·· for ≤≤ and ≥≥ inequalities, and use open circles ∘∘ for $$ inequalities.

4. Test each interval defined by the critical values. If an interval satisfies the inequality, then it is part of the solution. If it does not satisfy the inequality, then it is not part of the solution.

Properties :

Inequalities are governed by the following properties. All of these properties also hold if all of the non-strict inequalities (≤ and ≥) are replaced by their corresponding strict inequalities (< and >) and (in the case of applying a function) monotonic functions are limited to strictly monotonic functions.

- **Transitivity** - The transitive property of inequality states:

For any real numbers a, b, c:

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If $a \geq b$ and $b \geq c$, then $a \geq c$.

If $a \leq b$ and $b \leq c$, then $a \leq c$.

If either of the premises is a strict inequality, then the conclusion is a strict inequality.

E.g. if $a \geq b$ and $b > c$, then $a > c$

An equality is of course a special case of a non-strict inequality.

E.g. if $a = b$ and $b > c$, then $a > c$

- **Converse** - The relations ≤ and ≥ are each other's converse:

For any real numbers a and b:

If $a \leq b$, then $b \geq a$.

If $a \geq b$, then $b \leq a$.

- **Addition and subtraction -** A common constant c may be added to or subtracted from both sides of an inequality:

For any real numbers a, b, c

If $a \leq b$, then $a + c \leq b + c$ and $a - c \leq b - c$.

If $a \geq b$, then $a + c \geq b + c$ and $a - c \geq b - c$.

i.e., the real numbers are an ordered group under addition.

- **Multiplication and division -** The properties that deal with multiplication and division state:

For any real numbers, a, b and non-zero c:

If c is positive, then multiplying or dividing by c does not change the inequality:

If $a \geq b$ and $c > 0$, then $ac \geq bc$ and $a/c \geq b/c$.

If $a \leq b$ and $c > 0$, then $ac \leq bc$ and $a/c \leq b/c$.

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If c is negative, then multiplying or dividing by c inverts the inequality:

If $a \geq b$ and $c < 0$, then $ac \leq bc$ and $a/c \leq b/c$.

If $a \leq b$ and $c < 0$, then $ac \geq bc$ and $a/c \geq b/c$.

More generally, this applies for an ordered field, see below.

- **Additive inverse**

The properties for the additive inverse state:

For any real numbers a and b, negation inverts the inequality:

If $a \leq b$, then $-a \geq -b$.

If $a \geq b$, then $-a \leq -b$.

- **Multiplicative inverse**

The properties for the multiplicative inverse state:

For any non-zero real numbers a and b that are both positive or both negative:

If $a \leq b$, then $1/a \geq 1/b$.

If $a \geq b$, then $1/a \leq 1/b$.

If one of a and b is positive and the other is negative, then:

If $a < b$, then $1/a < 1/b$.

If $a > b$, then $1/a > 1/b$.

These can also be written in chained notation as:

For any non-zero real numbers a and b:

If $0 < a \leq b$, then $1/a \geq 1/b > 0$.

If $a \leq b < 0$, then $0 > 1/a \geq 1/b$.

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If $a < 0 < b$, then $1/a < 0 < 1/b$.

If $0 > a \geq b$, then $1/a \leq 1/b < 0$.

If $a \geq b > 0$, then $0 < 1/a \leq 1/b$.

If $a > 0 > b$, then $1/a > 0 > 1/b$.