Topic: Permutation And Combination

INTRODUCTION

PERMUTATIONS :

Suppose you want to arrange your books on a shelf. If you have only one book, there is only one way of arranging it. Suppose you have two books, one of History and one of Geography.

You can arrange the Geography and History books in two ways. Geography book first and the History book next, GH or History book first and Geography book next; HG. In other words, there are two arrangements of the two books.

Now, suppose you want to add a Mathematics book also to the shelf. After arranging History and Geography books in one of the two ways, say GH, you can put Mathematics book in one of the following ways: MGH, GMH or GHM. Similarly, corresponding to HG, you have three other ways of arranging the books. So, by the Counting Principle, you can arrange Mathematics, Geography and History books in $3\times2=6$ ways.
By permutation we mean an arrangement of objects in a particular order.

In general, if you want to find the number of permutations of n objects $n\geq1$, how can you do it? Let us see if we can find an answer to this.

Similar to what we saw in the case of books, there is one permutation of 1 object, 2×1 permutations of two objects and $3\times2\times1$ permutations of 3 objects. It may be that, there are $n\times(n-1)\times(n-2)\times\ldots\times2\times1$ permutations of n objects.

"The different arrangements of a given number of things by taking some or all at a time, are called permutations."

Examples:

i. All permutations (or arrangements) made with the letters *a*, *b*, *c* by taking two at a time are (*ab*, *ba*, *ac*, *ca*, *bc*, *cb*).
ii. All permutations made with the letters *a*, *b*, *c* taking all at a time are: (*abc*, *acb*, *bac*, *bca*, *cab*, *cba*)

Number of Permutations:

Number of all permutations of *n* things, taken *r* at a time, is given by:

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Examples:

i. $^6P_2 = (6 \times 5) = 30$.
ii. $^7P_3 = (7 \times 6 \times 5) = 210$.
iii. Cor. number of all permutations of n things, taken all at a time = n! .

An Important Result:

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If there are n subjects of which p_1 are alike of one kind; p_2 are alike of another kind; p_3 are alike of third kind and so on and p_r are alike of r^{th} kind,
such that $(p_1 + p_2 + \dots p_r) = n$.

Then, number of permutations of these n objects is = $\frac{n!}{(p_1!).(p_2)!.....(p_r!)}$

Factorial Notation:

Since the definition or the formula of both, permutation and combination, requires the use of factorial notation, so let's first understand this here before learning any further.

In Mathematics, the factorial is represented by the symbol '!' i.e. if we have to write 5 factorial, so it will be written as 5! So in general factorial of any positive number n will be represented by n!.

Mathematically,

$$n! = \begin{cases} 1 & if\ n = 0 \\ (n-1)!\ n & if\ n > 0 \end{cases}$$

where n is any positive integer.

So, 4! = 3! x 4 = 2! x 3 x 4 = 1! x 2 x 3 x 4 = 0! x 1 x 2 x 3 x 4 = 1 x 2 x 3 x 4

Similarly we can say for any positive integer 'n'

n! = n x (n – 1) x (n – 2) x …….. x 3 x 2 x 1.

Thus, we seeing the above equation, we may also define the factorial of any positive integer n as 'the product of all the positive integers less than or equal to n'.

Just see below for the factorial of few frequently used numbers.

0! = 1

1! = 1

2! = 2 x 1 = 2

3! = 3 x 2 x 1 = 6

4! = 4 x 3 x 2 x 1 = 24

5! = 5 x 4 x 3 x 2 x 1 = 120 and so on.

Let n be a positive integer. Then, factorial n, denoted $n!$ is defined as: **n! = n(n - 1)(n - 2) ... 3.2.1.**

Examples:

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i. We define **0! = 1**.
ii. 4! = (4 x 3 x 2 x 1) = 24.
iii. 5! = (5 x 4 x 3 x 2 x 1) = 120.

Permutation of R objects Out of N objects :

Suppose you have five story books and you want to distribute one each to Asha, Akhtar and Jasvinder. In how many ways can you do it? You can give any one of the five books to Asha and after that you can give any one of the remaining four books to Akhtar. After that, you can give one of the remaining three books to Jasvinder. So, by the Counting Principle, you can distribute the books in 5×4×3 i.e. 60 ways.
More generally, suppose you have to arranger objects out of n objects. In how many ways can you do it? Let us view this in the following way. Suppose you haven objects and you have to arrange r of these in r boxes, one object in each box.

| n ways| | n-1 ways | .... | n-r+1 ways |
___________________ r boxes______________________

Suppose there is one box. r=1. You can put any of the n objects in it and this can be done in n ways. Suppose there are two boxes. r=2. You can put any of the objects in the first box and after that the second box can be filled with any of the remaining n−1 objects. So, by the counting principle, the two boxes can be filled in n(n−1) ways.
Similarly, 3 boxes can be filled in n(n−1)(n−2) ways.

Examples:
If you have 6 New Year greeting cards and you want to send them to 4 of your friends, in how many ways can this be done?

Solution:

We have to find number of permutations of 4 objects out of 6 objects.

This number is $^6P_4=6(6-1)(6-2)(6-3)=6\times5\times4\times3=360$
Therefore, cards can be sent in 360 ways.

So, using the factorial notation, this formula can be written as follows:

$^nP_r=n! / (n-r)!$

Permutations under Some Conditions :

- Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement is: $r\,^{n-1}Pr-$
- Number of permutations of n different things, taken r at a time, when a particular thing is never taken n each arrangement is: ^{n-1}Pr
- Number of permutations of n different things, taken all at a time, when m specified things always come together is: $m!\times(n-m+1)!$
- Number of permutations of n different things, taken all at a time, when m specified never come together is: $n!-[m!\times(n-m+1)!]$

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- The number of permutations of n dissimilar things taken r at a time when k(< r) particular things always occur is: $[^{n-k}P_{r-k}]\times[^{r}P_{k}]$x
- The number of permutations of n dissimilar things taken r at a time when k particular things never occur is: ^{n-1}Pr
- The number of permutations of n dissimilar things taken r at a time when repetition of things is allowed any number of times is: n^r
- The number of permutations of n different things, taken not more than r at a time, when each thing may occur any number of times is: $n+n^2+n^3+....+n^r= n(n^r-1) / (n-1)$
- The number of permutations of n different things taken not more than r at a time: $^nP_1+^nP_2+^nP_3+...+^nP_r$

COMBINATIONS :

Let us consider the example of shirts and trousers as stated in the introduction. There you have 4 sets of shirts and trousers and you want to take 2 sets with you while going on a trip. In how many ways can you do it?

Let us denote the sets by S1,S2,S3,S4. Then you can choose two pairs in the following ways:
1. {S1,S2}
2. {S1,S3}
3. {S1,S4}
4. {S2,S3}
5. {S2,S4}
6. {S3,S4}

[Observe that {S1,S2} is the same as {S2,S1}. So, there are 6 ways of choosing the two sets that you want to take with you. Of course, if you had 10 pairs and you wanted to take 7 pairs, it will be much more difficult to work out the number of pairs in this way.
Now as you may want to know the number of ways of wearing 2 out of 4 sets for two days, say Monday and Tuesday, and the order of wearing is also important to you. We know that it can be done in $^4P_4$=12 ways. But note that each choice of 2 sets gives us two ways of wearing 2 sets out of 4 sets as shown below:

1. {S1,S2} -> S1 on Monday and S2 on Tuesday or S2 on Monday and S1 on Tuesday
2. {S1,S3} -> S1 on Monday and S3 on Tuesday or S3 on Monday and S1 on Tuesday
3. {S1,S4} -> S1 on Monday and S4 on Tuesday or S4 on Monday and S1 on Tuesday
4. {S2,S3} -> S2 on Monday and S3 on Tuesday or S3 on Monday and S2 on Tuesday
5. {S2,S4} -> S2 on Monday and S4 on Tuesday or S4 on Monday and S2 on Tuesday
6. {S3,S4} -> S3 on Monday and S4 on Tuesday or S4 on Monday and S3 on Tuesday

Thus, there are 12 ways of wearing 2 out of 4 pairs.

"Each of the different groups or selections which can be formed by taking some or all of a number of objects is called a combination."

Examples:

1. Suppose we want to select two out of three boys A, B, C. Then, possible selections are AB, BC and CA.

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Note: AB and BA represent the same selection.

2. All the combinations formed by *a*, *b*, *c* taking ***ab***, ***bc***, ***ca***.
3. The only combination that can be formed of three letters *a*, *b*, *c* taken all at a time is ***abc***.
4. Various groups of 2 out of four persons A, B, C, D are: **AB, AC, AD, BC, BD, CD**.
5. Note that *ab ba* are two different permutations but they represent the same combination.

Number of Combinations:

The number of all combinations of *n* things, taken *r* at a time is:

$$^nC_r = \frac{n!}{(r!)(n-r)!} = \frac{n(n-1)(n-2)\ \dots\ \text{to } r \text{ factors}}{r!}.$$

Note:

i. $^nC_n = 1$ and $^nC_0 = 1$.

ii. $^nC_r = {^nC_{(n-r)}}$

Examples:

i. $^{11}C_4 = \frac{(11 \times 10 \times 9 \times 8)}{(4 \times 3 \times 2 \times 1)} = 330.$

ii. $^{16}C_{13} = {^{16}C_{(16-13)}} = {^{16}C_3} = \frac{16 \times 15 \times 14}{3!} = \frac{16 \times 15 \times 14}{3 \times 2 \times 1} = 560.$

Permutation Vs Combination

The word **permutation** means **arrangement** of the alike or different objects taken some or all at a time. So we can observe the word 'arrangement' used in the definition of permutation. Here the arrangement means selection as well as ordering. That means the order in which the objects are selected have also been taken care of in this case.

Example – The number of 5 digit numbers which can be formed using the digits 0, 1, 2, 3, 4 and 5.

In this example, we just not have to select the 5 digits out of given 6 digits but also have to see the number of possible cases for the different arrangement. So the numbers 34251, 21034, 42351 are all different cases.

The very basic difference in permutation and combination is the **order** of the objects considered. In combination, the order is not considered at all while for permutation it is must. So the permutation is the ordered arrangement while the combination is the unordered selection.

From the three alphabets A, B and C, the permutation of these 3 letters will be ABC, ACB, BAC, BCA, CBA and CAB. While the combination of 3 letters will be just (A, B, C).

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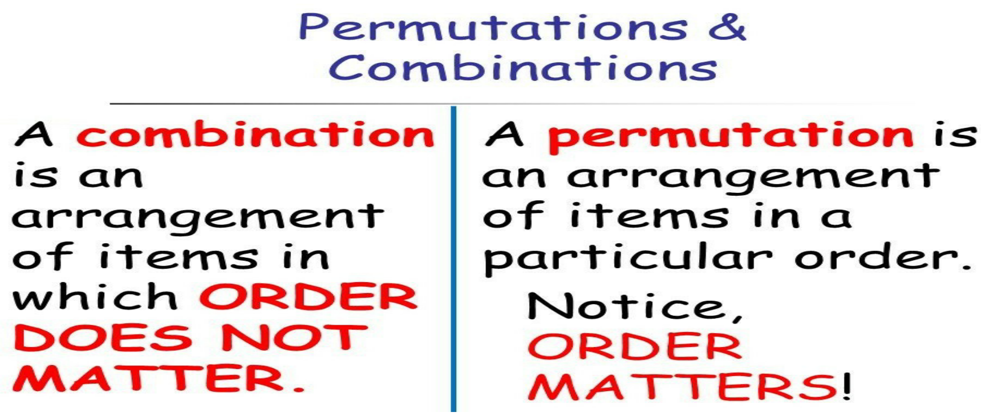

Permutation gives the answer to the number of arrangements while the combination explains the possible number of selections.

Permutation of a single combination can be multiple but the combination of a single permutation is unique (considering all at a time).

Relation between Permutation and Combination :

As discussed in the previous sections, permutation is the combination (or selection) and the arrangement as well. Thus, while calculating the permutation, we first need to choose or selecting the thing before their arrangement. So, **Permutation = Selection x Arrangement**

This can also be understood from their mathematical relation. Since we know that,

$$^nP_r = \frac{n!}{(n-r)!} \quad \text{and} \quad ^nC_r = \frac{n!}{(n-r)!\,r!}$$

Thus, from the above two formulas, this is very clear that

$$\mathbf{^nP_r = {^nC_r}\, r!}$$

where, ^nC_r denotes the selection and **r!** denotes the arrangement of r objects for the r places.